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Slide of the Seminar

Clustering of particles falling in a random flow

Dr. Kristian Gustafsson

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(P.I. Prof. Luca Biferale)

Università degli Studi di Roma Tor Vergata
C.F. n. 80213750583 – Partita IVA n. 02133971008 - Via della Ricerca Scientifica, 1 – 00133 ROMA

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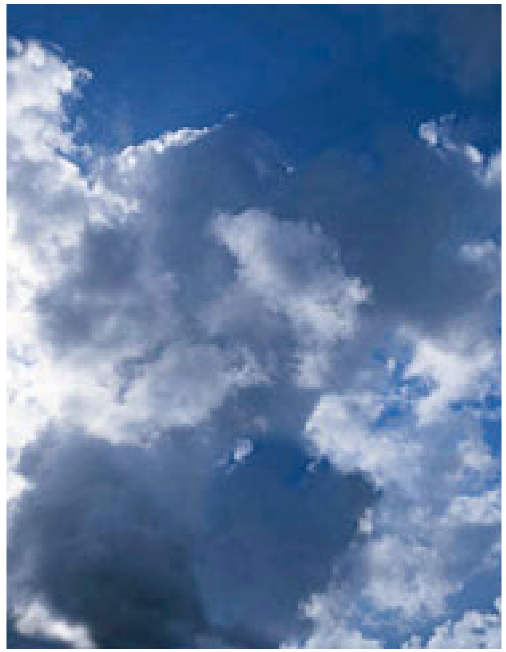
K. Gustavsson, B. Mehlig, S. Vajedi arXiv:1401.0513

kristian.gustafsson@physics.gu.se

Department of physics
University of Gothenburg

Water droplets in turbulent rain clouds



Forces on small droplet

Gravity (Newton's second law):

$$\boldsymbol{F}_{\mathrm{G}} = m\,\boldsymbol{g} = \frac{4\pi\rho_{\mathrm{p}}}{3}a^3\boldsymbol{g}$$

ρ_{p}	density of water droplet
g	gravitational acceleration
a	particle size

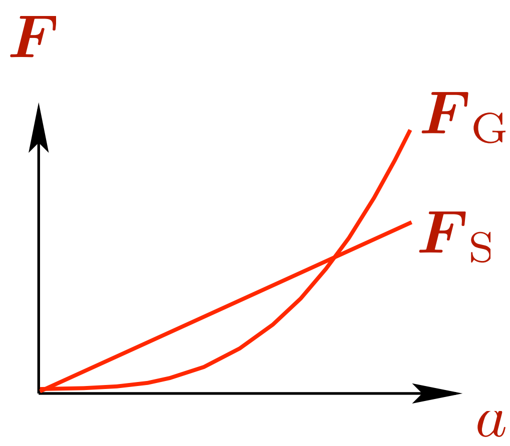
Friction (Stokes' law):

$$\boldsymbol{F}_{\mathrm{S}} = \mu\,\big(\boldsymbol{u}(\boldsymbol{r},\mathrm{t}) - \boldsymbol{v}\big)$$

where

$$\mu = 6\pi\rho_{\mathrm{p}}\nu a \quad (= m\gamma\,)$$

ν	viscosity
$\boldsymbol{u}(\boldsymbol{r},\mathrm{t})$	velocity of turbulent air in cloud
$\boldsymbol{r}$	droplet position
$\boldsymbol{v}$	droplet velocity



Model

Spherical droplets move independently

Particle equation of motion

$$\dot{\boldsymbol{r}} = \boldsymbol{v}$$

$$\dot{\boldsymbol{v}} = \gamma(\boldsymbol{u}(\boldsymbol{r},t) - \boldsymbol{v}) + \boldsymbol{g}$$

γ damping rate (depends on droplet size and mass)

$\boldsymbol{r}$ particle position

$\mathbf{v}$ particle velocity

$\boldsymbol{g}$ gravitational acceleration (or a mean flow)

$\mathbf{u}(\mathbf{r},t)$ stationary incompressible random velocity field
no preferred direction or position in either space or time
single scale flow with typical length scale η , time scale τ and speed u_0

$$\langle u(\boldsymbol{x}_1,t)\rangle = 0$$

$$\langle u(\boldsymbol{x}_1,t_1)u(\boldsymbol{x}_2,t_2)\rangle \sim u_0^2 e^{-|t_1-t_2|/\tau-(\boldsymbol{x}_1-\boldsymbol{x}_2)^2/(2\eta^2)}$$

Question: How do particles cluster within this model?

Model parameters

- u_0 flow speed
- η correlation length of flow
- γ damping rate
- τ correlation time of flow
- g gravitational acceleration

Dimensionless parameters:

Kubo number $\mathrm{Ku} = u_0\tau/\eta$

Stokes number $\mathrm{St} = 1/(\gamma\tau)$

Dimensionless gravity $F = g\tau/u_0$

In rain cloud turbulence:

$$\mathrm{Ku} \sim 1$$

$$F \sim 1$$

$$\mathrm{St} \sim 10^9 a^2$$

(a particle size in meter)

R. Shaw, Annu. Rev. Fluid Mech **35** (2003)

Small droplet

$a = 1\,\mu\mathrm{m}$

$\mathrm{St} \sim 10^{-3}$

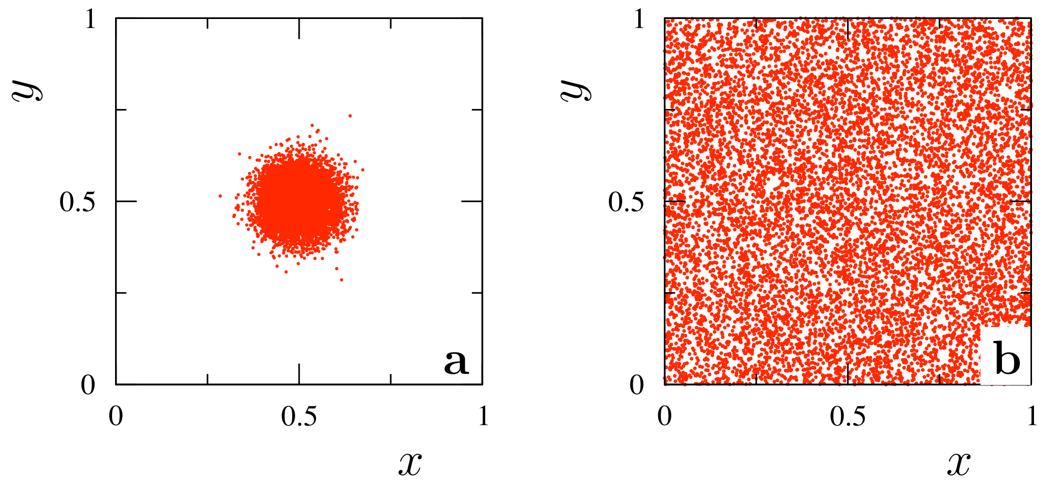
Large droplet

$a = 100\,\mu\mathrm{m}$

$\mathrm{St} \sim 10$

Mixing by random stirring

Computer simulation of 10^4 particles (red) in two-dimensional random flow (periodic boundary conditions in space)



a initial distribution, **b** particle positions after random stirring.

'Unmixing' of slightly inertial particles

Non-interacting, non-colliding particles (red) suspended in a random flow

Particle equation of motion

$$\dot{\boldsymbol{v}} = \frac{1}{\mathrm{St}}(\boldsymbol{u}(\boldsymbol{r},t) - \boldsymbol{v})$$

$$\mathrm{St} = 0.1$$

$$\mathrm{Ku} = 1$$

$$F = 0$$


Region of high vorticity


Particle density

Maxey centrifuge effect Maxey, J. Fluid Mech. **174**, 441, (1987)

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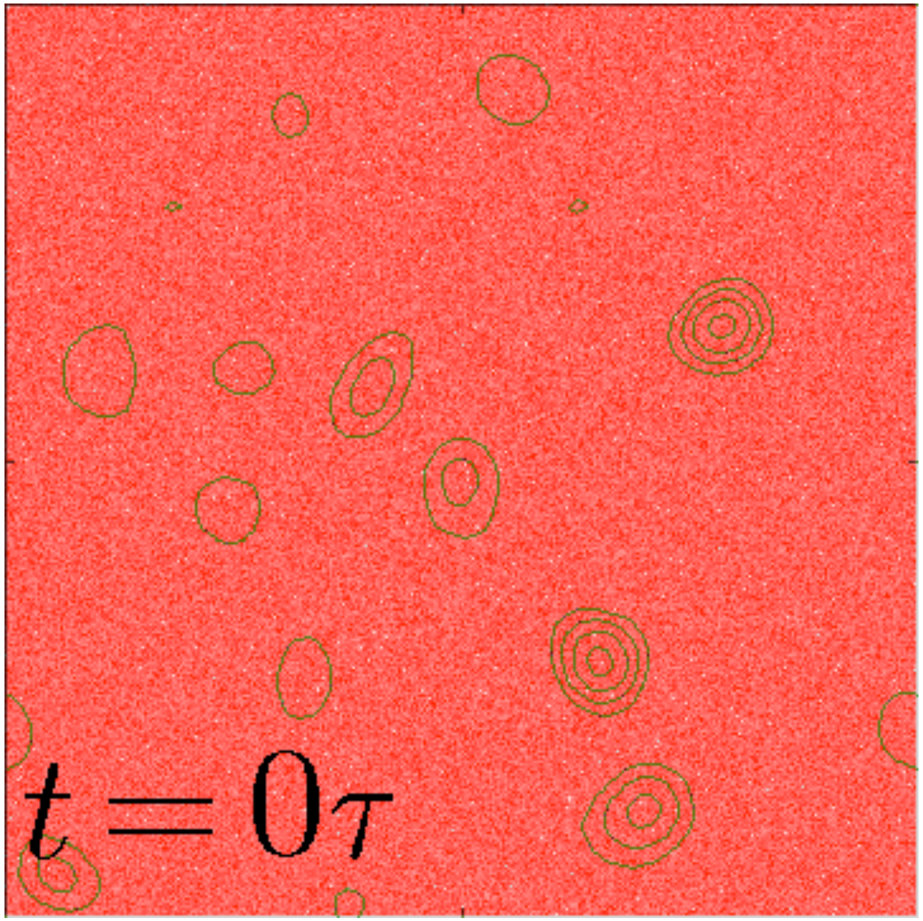


Maxey centrifuge effect Maxey, J. Fluid Mech. **174**, 441, (1987)

Preferential concentratration

Maxey, J. Fluid Mech. **174**, 441, (1987)

Droplets are centrifuged away from vortices.

For slightly inertial particles ($\mathrm{St} \approx 0$)

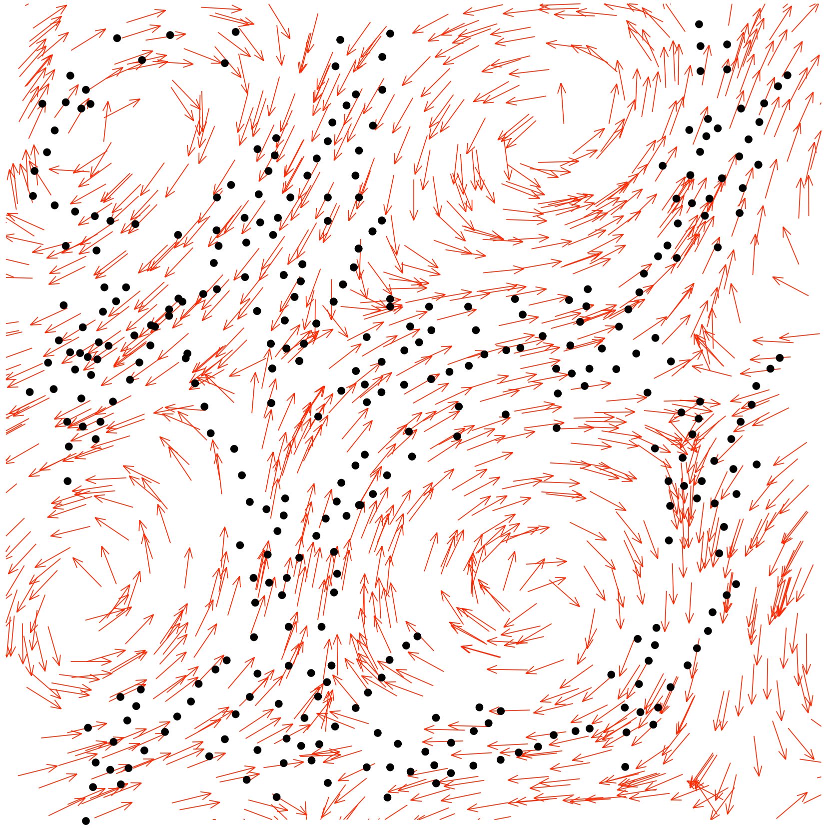
$$\mathbf{v} = \mathbf{u} - \mathrm{St}\left[\frac{\partial \mathbf{u}}{\partial t} + \mathrm{Ku}(\mathbf{u}\cdot\nabla)\mathbf{u}\right]$$

Particles follows effective velocity field $\mathbf{v}$, which is compressible

$$\nabla\cdot\mathbf{v} = -\mathrm{Ku}\,\mathrm{St}\,\mathrm{Tr}\left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^2\right]$$

$$= -\mathrm{Ku}\mathrm{St}[\mathrm{Tr}(\boldsymbol{S}^T\boldsymbol{S}) - \mathrm{Tr}(\boldsymbol{R}^T\boldsymbol{R})]$$

$\boldsymbol{S}$ Strain-rate, $\boldsymbol{R}$ Rotational part



Particles avoid regions of high vorticity and gather in regions of high strain.

Clustering because $\nabla\cdot\mathbf{v} < 0$ for typical trajectories.

‘Unmixing’ of very inertial particles

Non-interacting, non-colliding particles (red) suspended in a random flow

Particle equation of motion

$$\dot{\boldsymbol{v}} = \frac{1}{\mathrm{St}}(\boldsymbol{u}(\boldsymbol{r},t) - \boldsymbol{v})$$

$$\mathrm{St} = 10$$

$$\mathrm{Ku} = 0.1$$

$$F = 0$$

 Region of high vorticity

 Particle density

Multiplicative amplification Mehlig & Wilkinson, PRL **92** (2004) 250602

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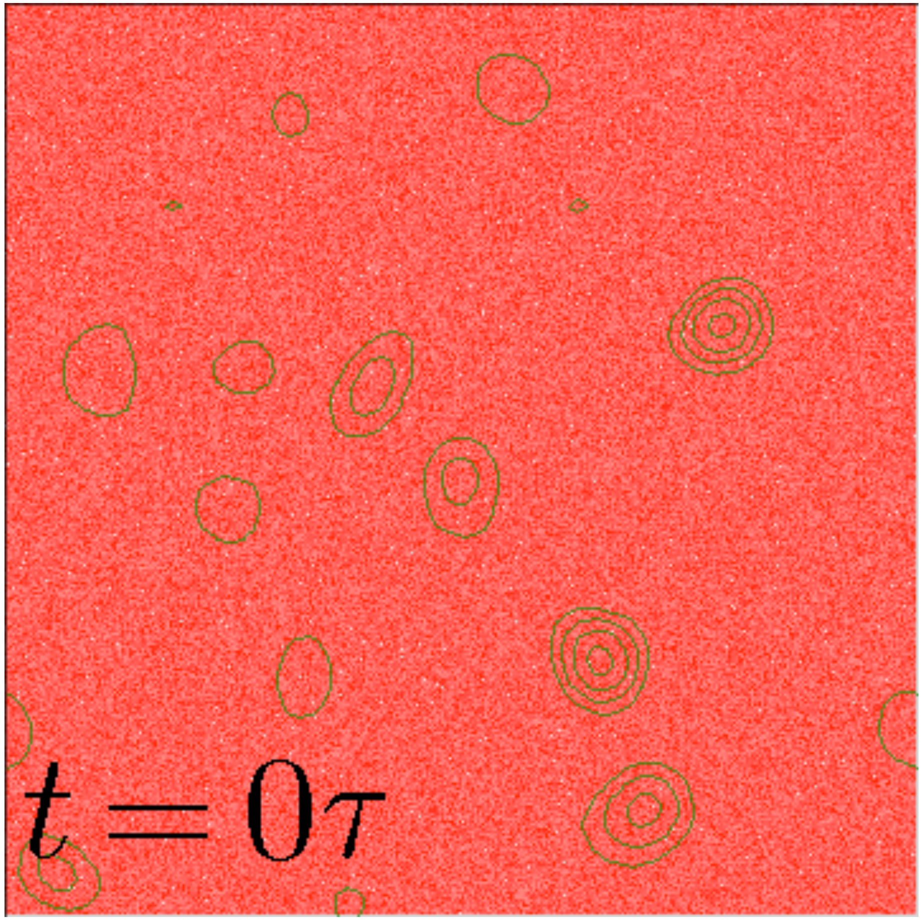


Multiplicative amplification Mehlig & Wilkinson, PRL **92** (2004) 250602

Multiplicative amplification

The motion of heavy particles ($\mathrm{St} \gg 1$) is independent of the instantaneous value of the force if Ku is small enough ($\mathrm{Ku} \ll \sqrt{\mathrm{St}}$).

Replace the position dependent $\mathbf{u}$ by ‘random kicks’:

$$\mathbf{u}(\mathbf{r}_t, t) \to \mathbf{u}(t)$$

Langevin/Fokker-Planck treatment possible.
Dynamics described by single parameter: $\epsilon^2 \sim \mathrm{Ku}^2\mathrm{St}$

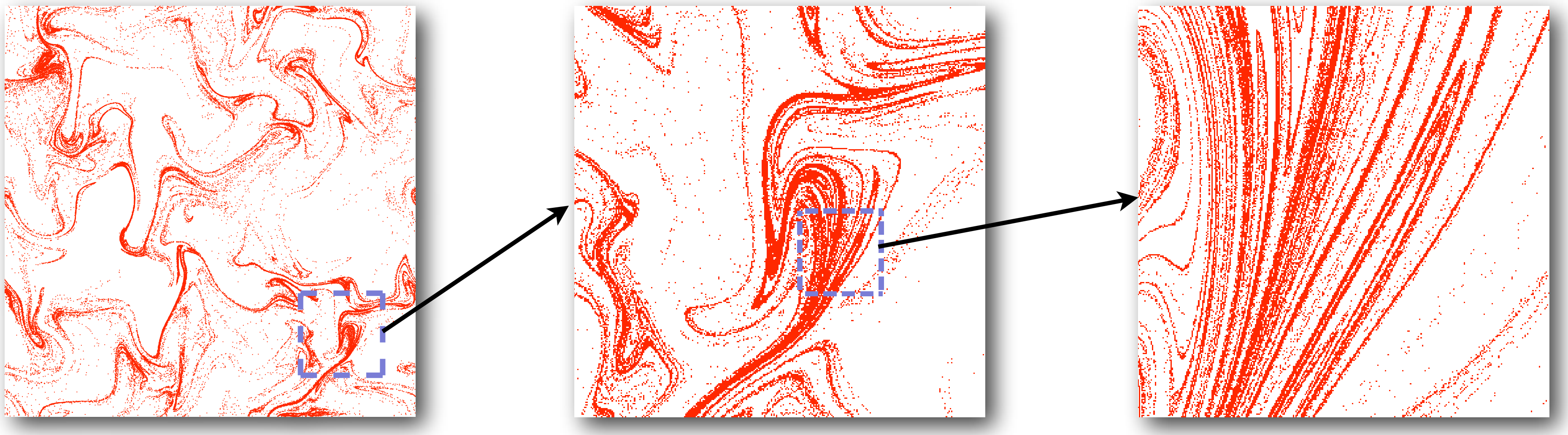
Clustering results as the net effect of many small deformations of particle velocity volumes, uncorrelated from any instantaneous structures in the flow.

Mehlig & Wilkinson, Phys. Rev. Lett. **92** (2004) 250602
Duncan et al., Phys. Rev. Lett. **95** (2005)
Wilkinson et al., Phys. Fluids **19** (2007) 113303

Fractal clustering

Particles cluster on self-similar structures, so called ‘fractals’

Sommerer & Ott, Science **259** , 334, (1993)



Fractal dimension somewhere between one and two

Quantification of clustering ($d = 2$)

Lyapunov exponents $\lambda_1 > \lambda_2$ describe rate of contraction or expansion of small length element δr_t , and area element $\delta \mathcal{A}_t$ of particle flow

$$\lambda_1 = \lim_{t\to\infty} t^{-1} \ln(\delta r_t)$$

$$\lambda_1 + \lambda_1 = \lim_{t\to\infty} t^{-1} \ln(\delta \mathcal{A}_t)$$

J. Sommerer & E. Ott, Science 259 (1993) 351

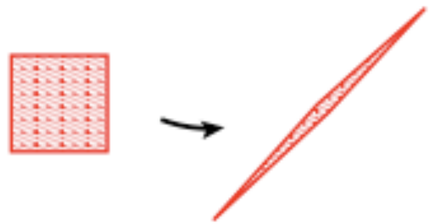
When $\mathrm{St} > 0$ and not too large, the dynamics is:

- chaotic (positive maximal Lyapunov exponent)

$$\lambda_1 > 0$$

- compressible (sum of two maximal Lyapunov exponents negative)

$$\lambda_1 + \lambda_2 < 0$$

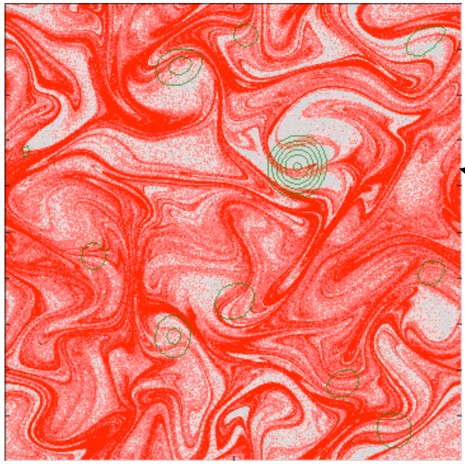


Fractal dimension $d_{\mathrm{L}} \equiv 2 - \frac{\lambda_1 + \lambda_2}{\lambda_2}$

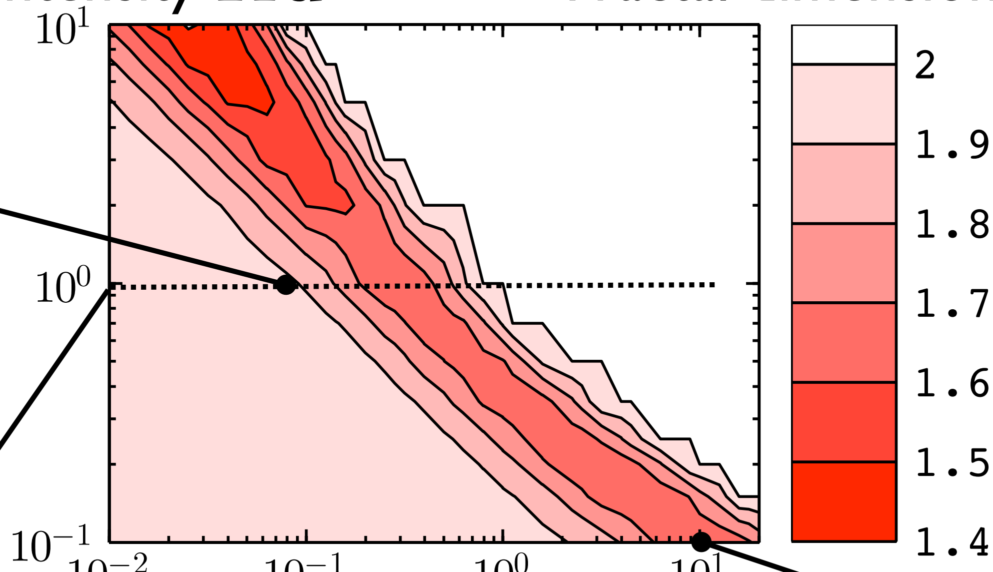
Kaplan & Yorke, Springer Lecture Notes in Mathematics **730**, 204, (1979)

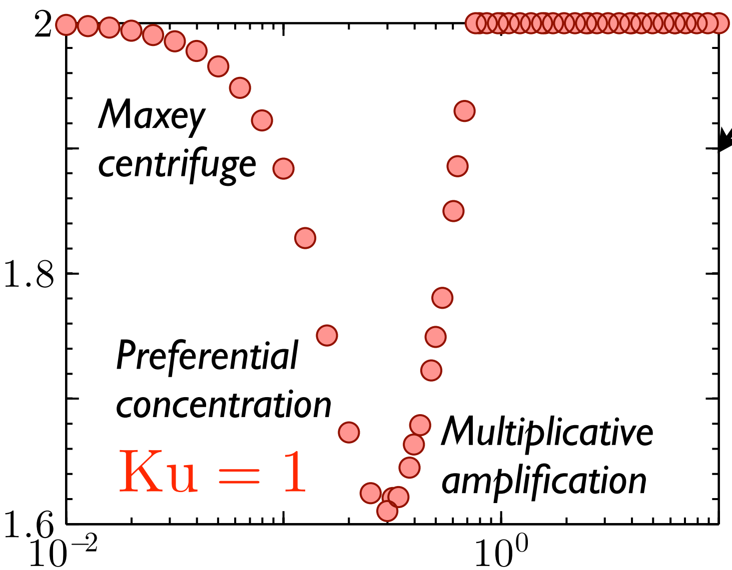
Clustering without gravity

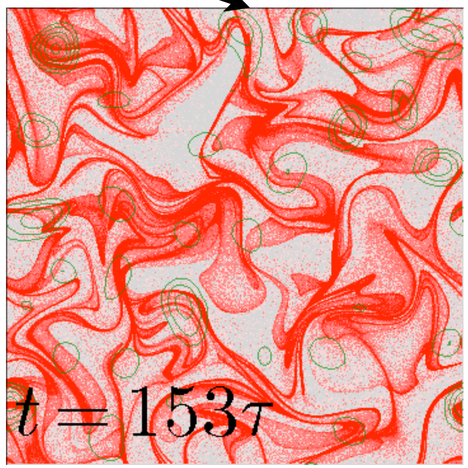
$$d_\mathrm{L} \equiv 2 - \frac{\lambda_1 + \lambda_2}{\lambda_2}$$



Maxey centrifuge effect







Multiplicative amplification

Deterministic dynamics with gravity

Dynamics in the absence of $\boldsymbol{u}$

$$\dot{\boldsymbol{r}} = \boldsymbol{v}$$
$$\dot{\boldsymbol{v}} = \gamma(\cancel{\boldsymbol{u}(\boldsymbol{r},t)} - \boldsymbol{v}) + \boldsymbol{g}$$

Deterministic solution

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_{\rm s}t + \gamma^{-1}(\boldsymbol{v}_0 - \boldsymbol{v}_{\rm s})(1 - e^{-\gamma t})$$
$$\boldsymbol{v} = \boldsymbol{v}_{\rm s} + (\boldsymbol{v}_0 - \boldsymbol{v}_{\rm s})e^{-\gamma t}$$

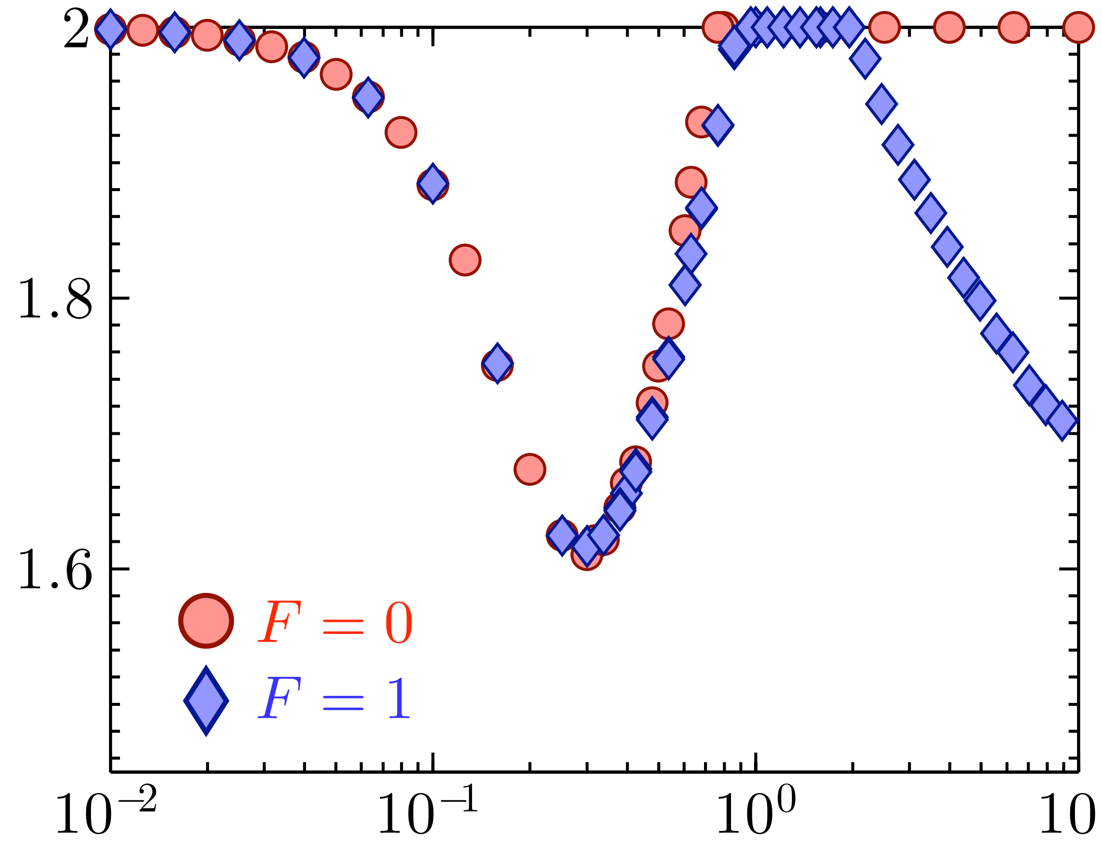
Particles reach a terminal 'settling velocity' $\boldsymbol{v}_{\rm s} \equiv \boldsymbol{g}/\gamma$

The deterministic solution is important if $v_{\rm s} \gg u_0$ ($F\mathrm{St} \gg 1$)

Relative motion between two particles is only affected by gravity through the $\boldsymbol{r}$-dependence in $\boldsymbol{u}(\boldsymbol{r},t)$. Gravity is expected to alter correlations between flow and particle trajectories.

Clustering with gravity ($\mathrm{Ku} = 1$)

Fractal dimension d_{L}



Inertia St

large $F\,\mathrm{St}$ (deterministic solution important)

‘Unmixing’ of falling inertial particles

Non-interacting, non-colliding particles (red) suspended in a random flow

$\mathrm{St} = 10$

$\mathrm{Ku} = 1$

$F = 1$

$\boldsymbol{v}_{\mathrm{s}}$

Frame moving with velocity $\boldsymbol{v}_{\mathrm{s}}$

Particle density

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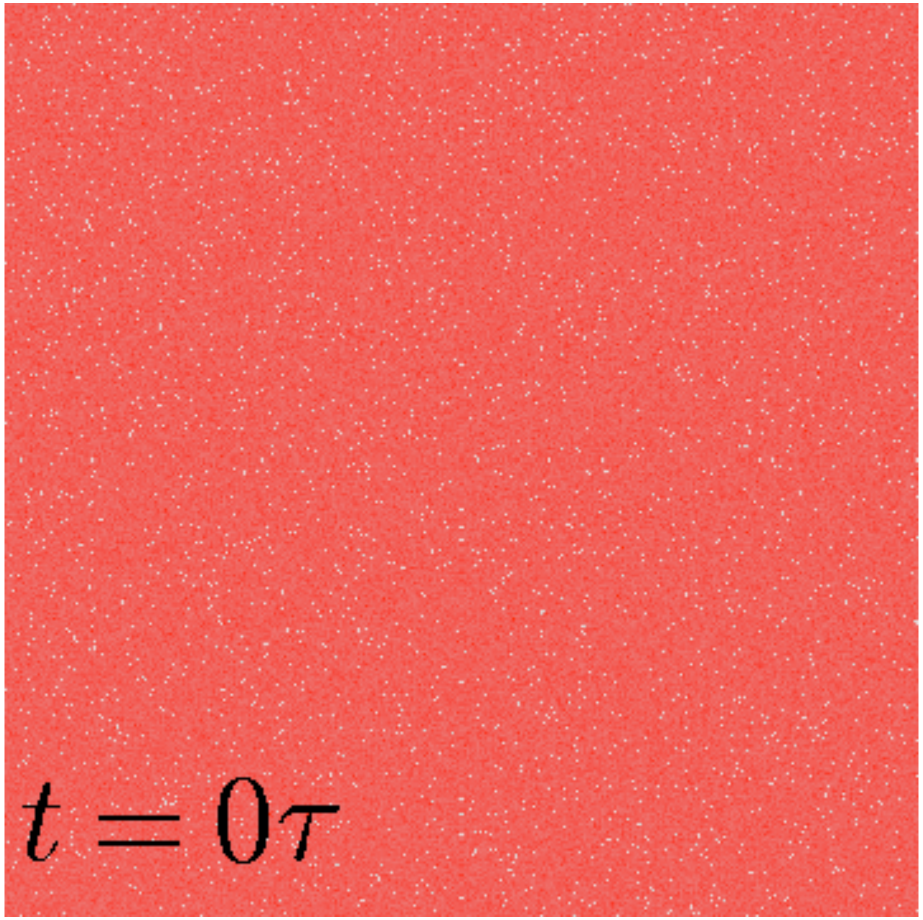
$F = 1$

Frame moving with velocity $\boldsymbol{v}_\mathrm{s}$

$\boldsymbol{v}_\mathrm{s}$



Large-St gravitational clustering

Particle density

Large-St dynamics

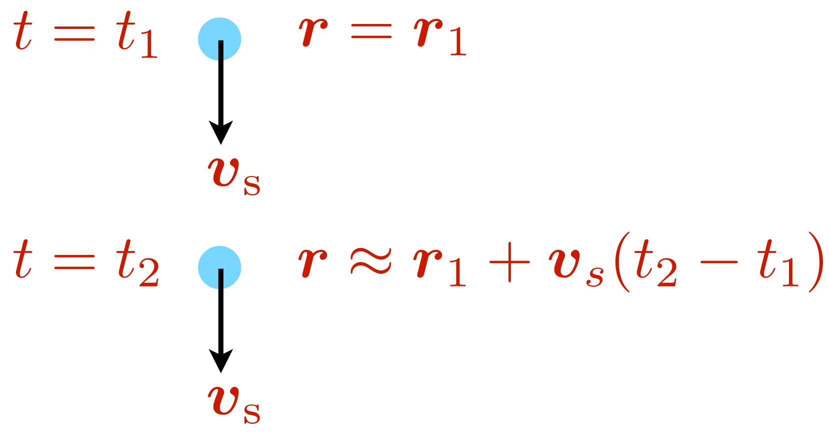
Deterministic solution $\boldsymbol{r} \approx \boldsymbol{r}_0 + \boldsymbol{v}_s t$ with settling velocity $\boldsymbol{v}_\mathrm{s} = \boldsymbol{g}/\gamma$

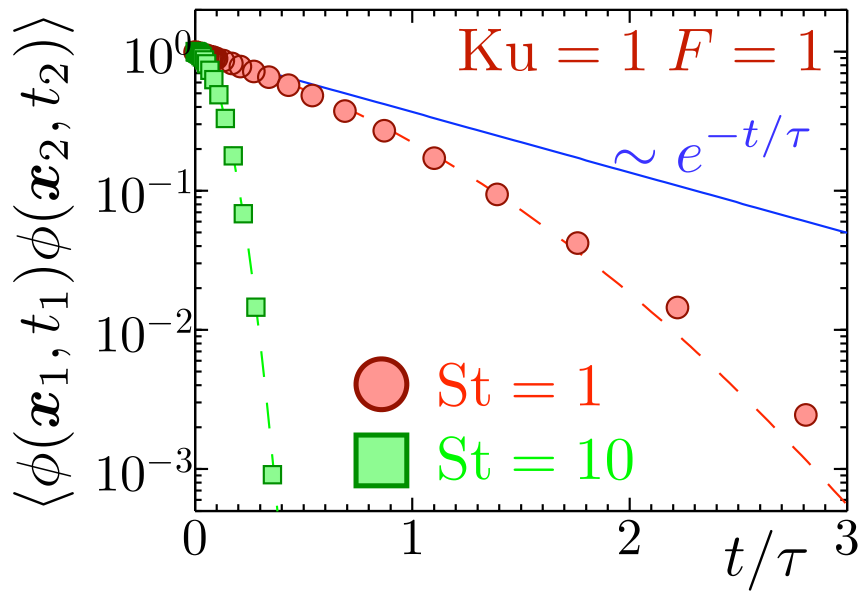
Spatial decorrelation becomes faster than time decorrelation.

Single-particle correlation function at two different times

$$\langle u(\boldsymbol{x}_1, t_1) u(\boldsymbol{x}_2, t_2)\rangle$$
$$\sim u_0^2 e^{-|t_1-t_2|/\tau - (\boldsymbol{x}_1-\boldsymbol{x}_2)^2/(2\eta^2)}$$
$$\sim u_0^2 e^{-|t_1-t_2|/\tau - v_\mathrm{s}^2(t_1-t_2)^2/(2\eta^2)}$$

When $G \equiv v_\mathrm{s}\tau/\eta = \mathrm{Ku}\,\mathrm{St}\,F$ is large the effective correlation time approaches white noise.





Langevin model

Langevin equation for separations $\boldsymbol{R}' = (\boldsymbol{r}_1 - \boldsymbol{r}_2)/\eta$ and relative velocities $\boldsymbol{V}' = (\boldsymbol{v}_1 - \boldsymbol{v}_2)/(\gamma\eta)$ ($t' = \gamma t$)

$$\delta \boldsymbol{R}' = \boldsymbol{V}'\,\delta t'\,, \qquad \delta \boldsymbol{V}' = -\boldsymbol{V}'\,\delta t' + \delta \boldsymbol{F}\,.$$

Increments $\delta\boldsymbol{F}$ are Gaussian white noise with $\langle \delta \boldsymbol{F}\rangle = \boldsymbol{0}$ and $\langle \delta F_i \delta F_j\rangle = 2\delta t' \mathrm{Ku}^2 \mathrm{St}\,\Sigma_{kl} D_{ik,jl} R'_k R'_l$ with $D_{ik,jl}$ obtained by integration of the effective correlation functions

$$D_{ik,jl} \equiv \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t \left\langle \frac{\partial u'_i}{\partial r'^k}(\boldsymbol{r}'(t'), t')\frac{\partial u'_j}{\partial r'^l}(\boldsymbol{0}, 0)\right\rangle \,.$$

We obtain ($\hat{\boldsymbol{g}} = -\boldsymbol{e}_y$)

$$D_{11,11} = D_{22,22} = -D_{11,22} = -D_{22,11} = -D_{12,21} = -D_{21,12} = \frac{1}{2G^2} - \frac{D_{21,21}}{3G^2}$$

$$D_{12,12} = \frac{G^2-1}{2G^4} + \frac{D_{21,21}}{3G^4}\,, \quad D_{21,21} = \frac{3}{\sqrt{8}G}\mathcal{F}\left[\frac{1}{\sqrt{2}G}\right], \mathcal{F}[x] \equiv \sqrt{\pi}e^{x^2}\mathrm{erfc}(x)\,.$$

Gravity introduces anisotropy ($D_{12,12} \neq D_{21,21}$)

Langevin model, large-G asymptote

Diagonalise and rescale noise

$$A_\pm \equiv \left(\frac{D_{21,21}}{D_{12,12}}\right)^{1/4} \frac{\partial u_1}{\partial r^2} \pm \left(\frac{D_{12,12}}{D_{21,21}}\right)^{1/4} \frac{\partial u_2}{\partial r^1}$$

For large values of $G = \mathrm{KuSt}F$ the dynamics is governed by a single parameter $D_{++} = D_{--} \sim \mathrm{Ku}^2\mathrm{St}/G^{3/2}$.

Compare this parameter to the parameter of the $F = 0$ white-noise model $\epsilon^2 \sim \mathrm{Ku}^2\mathrm{St}$.

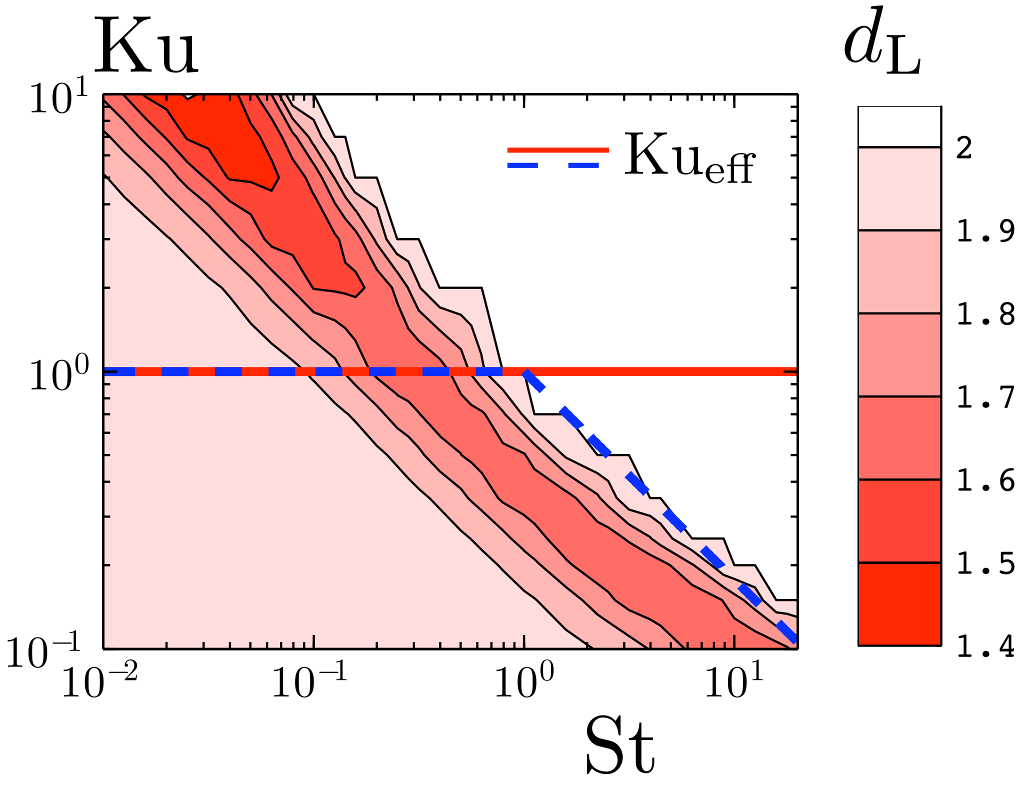
For a given large value of St define an effective Kubo number $\mathrm{Ku}_{\mathrm{eff}}$ in ϵ^2 so that the two parameters are equal

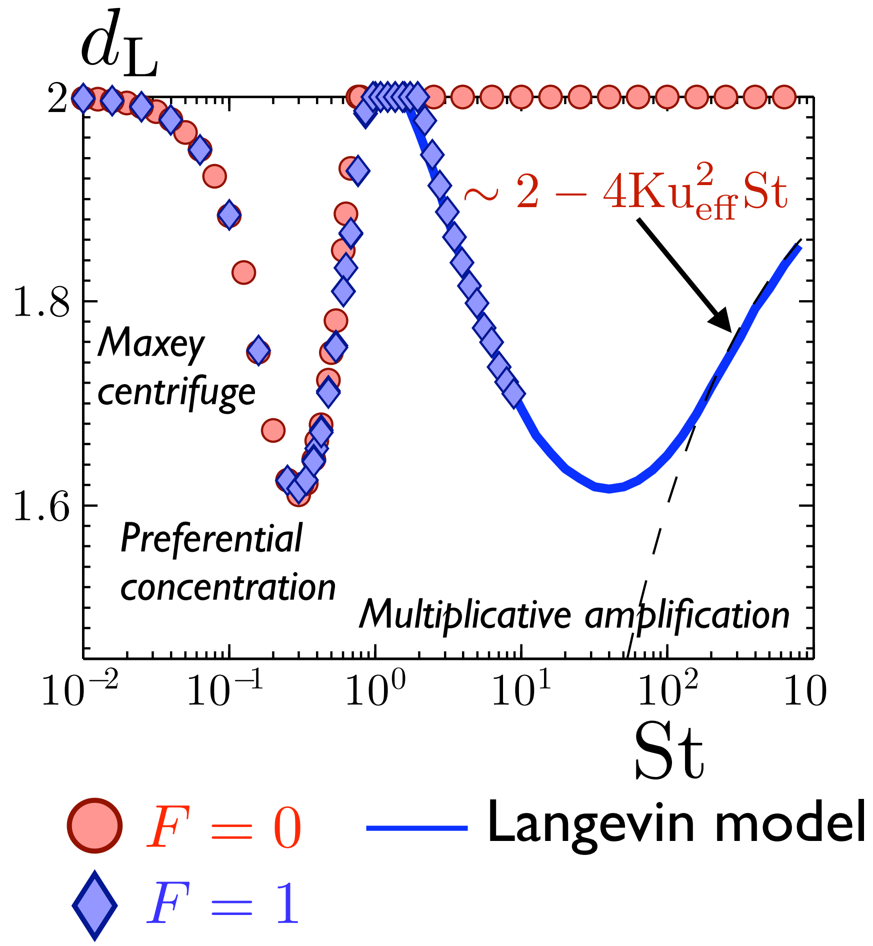
$$\mathrm{Ku}_{\mathrm{eff}} \sim \begin{cases} \mathrm{Ku} & \mathrm{St\ small} \\ \mathrm{Ku}^{1/4}/(F\mathrm{St})^{3/4} & \mathrm{St\ large} \end{cases} .$$

$\mathrm{Ku}_{\mathrm{eff}}$ approximately maps the $F \neq 0$ model with some value of Ku onto the $F = 0$ model with Kubo number $\mathrm{Ku}_{\mathrm{eff}}$.

Large-St gravitational clustering

The effective $\mathrm{Ku}_{\mathrm{eff}}$ maps the dynamics with $F > 0$, $\mathrm{Ku} = 1$ and large St on the $F = 0$ -dynamics

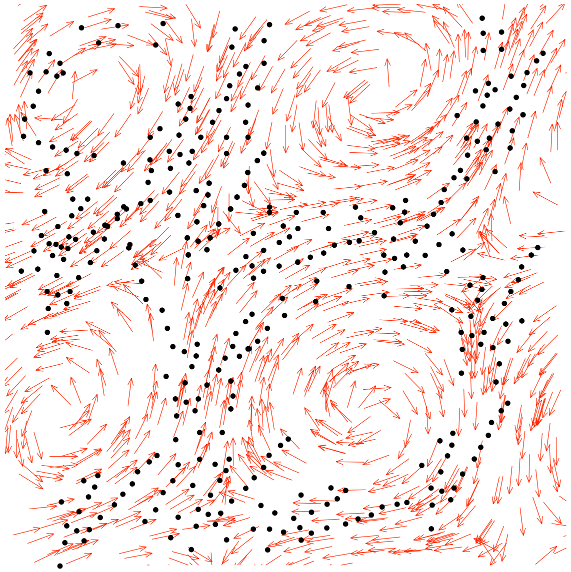




Clustering due to preferential sampling

As we have seen, gravity tends to enhance clustering due to multiplicative amplification for large values of St .

What is the effect of gravity on preferential sampling (e.g. Maxey centrifuge effect) and anisotropy for general values of F ?



+ gravity = ?

To answer this question we make a series expansion around deterministic trajectories.

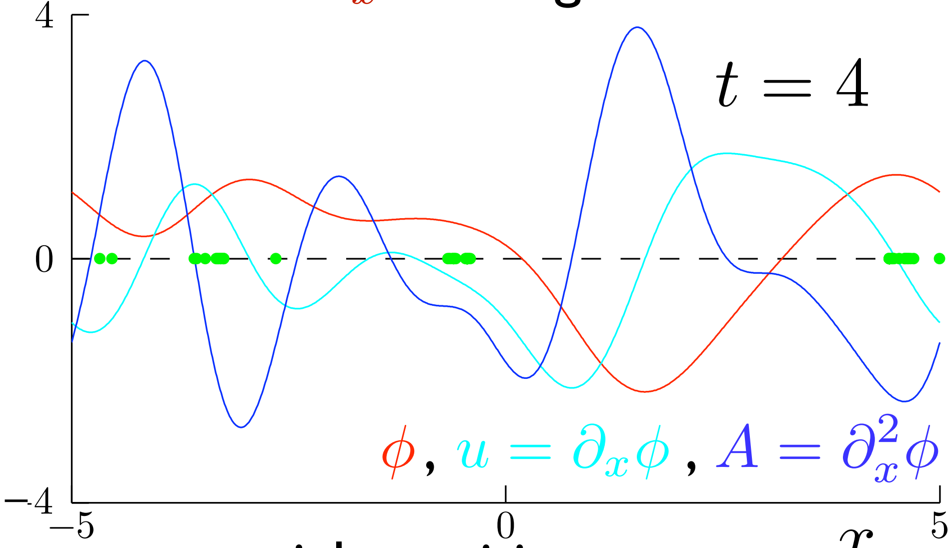
Example: preferential sampling ($d = 1$)

Particle following flow ($\mathrm{St} = 0$)

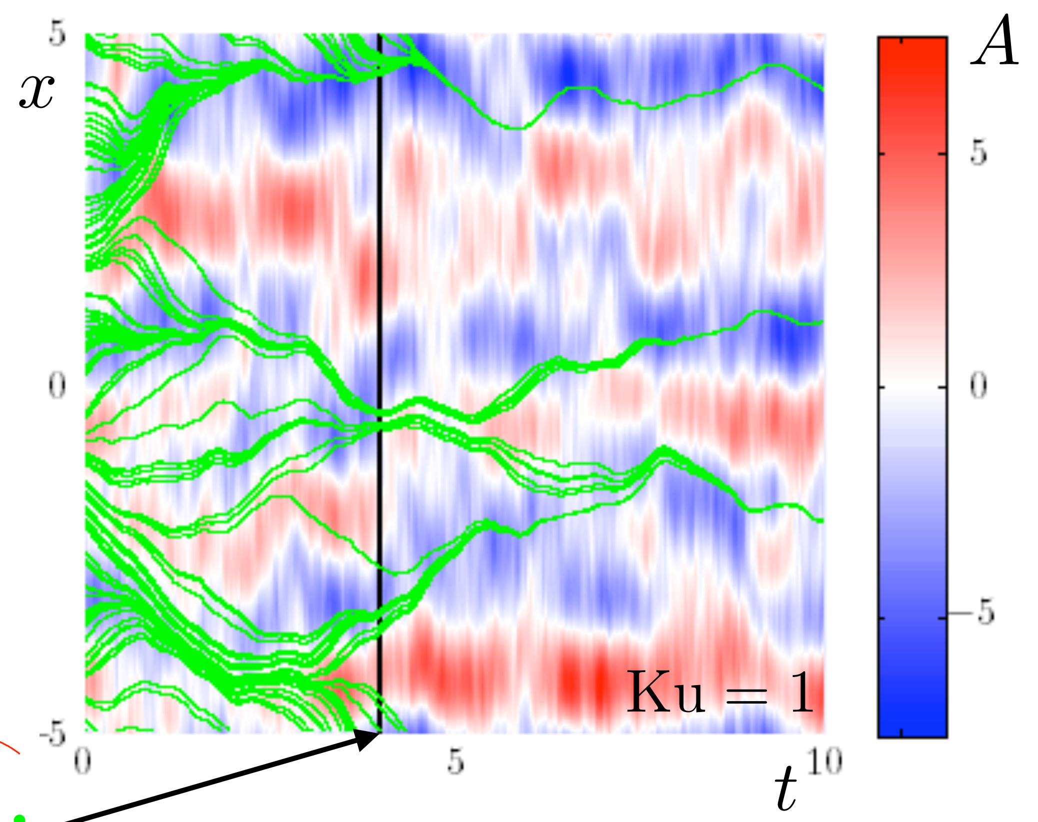
$$\dot{x}_t = u(x_t, t)$$

in a one-dimensional potential flow $u = \partial_x \phi$, with $\langle \phi(x,t) \rangle = 0$.

Particles tend to move towards potential maxima where the strain rate $A \equiv \partial_x u$ is negative.



● particle positions



A is non-ergodic: $\langle A(x_0, t) \rangle = 0$

$$\overline{A(x_t, t)} \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t \, A(x_t, t) < 0$$

Trajectory approximation ($d = 1$)

Start from dimensionless equations of motion

$$\ddot{x}_t = (u(x_t, t)\mathrm{Ku} - \dot{x}_t)/\mathrm{St}$$

Implicit solution

$$x_t = \tilde{x}_t + \frac{\mathrm{Ku}}{\mathrm{St}} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 e^{-(t_1 - t_2)/\mathrm{St}} u(x_{t_2, t_2}) \qquad \text{(i)}$$

with deterministic part $\tilde{x}_t = x_0 + \mathrm{St}(1 - e^{-t/\mathrm{St}})\dot{x}_0$.

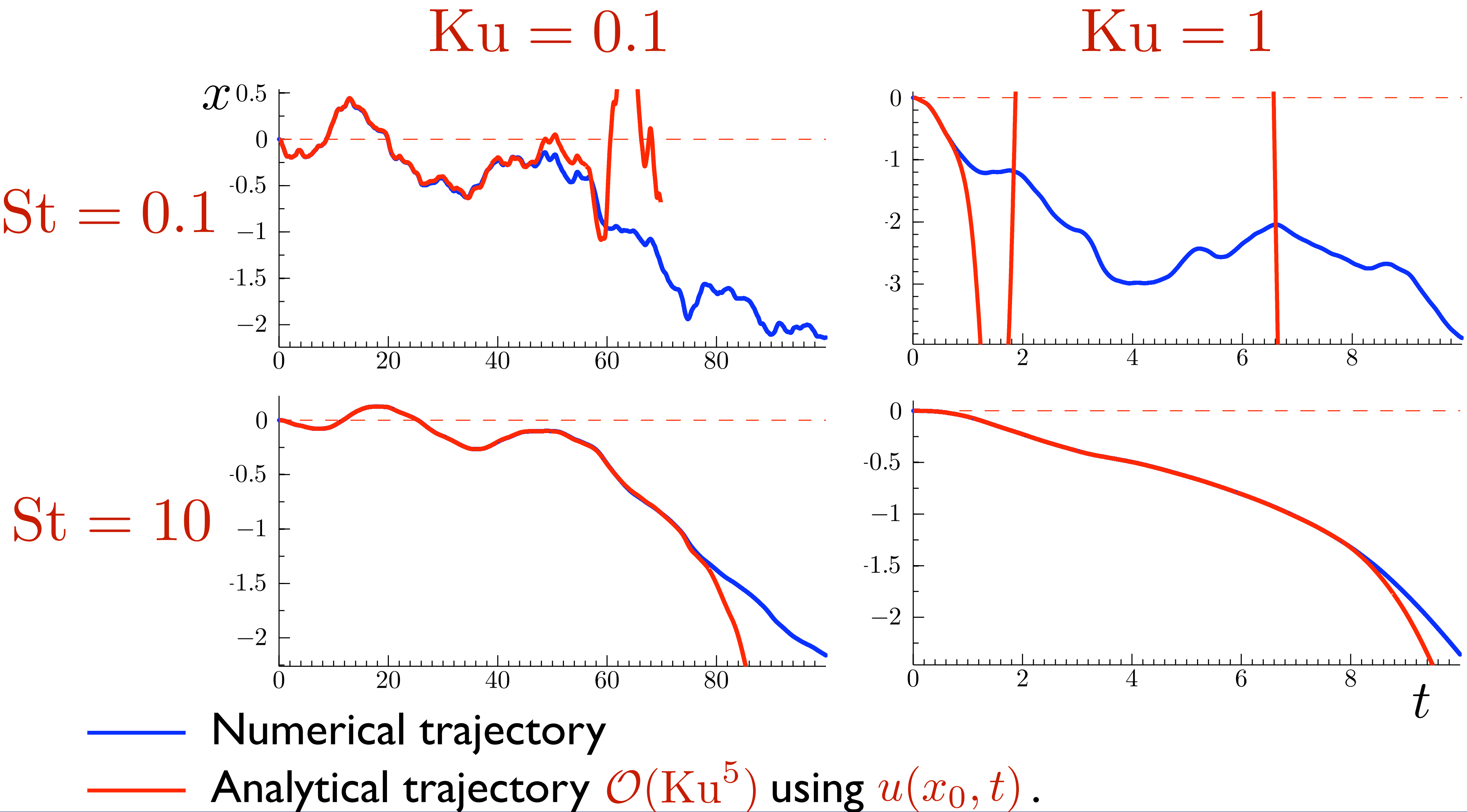
Assume $|x_t - \tilde{x}_t|$ small for all times up to t and expand $u(x_t, t)$ around $\tilde{x}_t$

$$u(x_t, t) = u(\tilde{x}_t, t) + \partial_x u(\tilde{x}_t, t)(x_t - \tilde{x}_t) + \frac{1}{2}\partial_x^2 u(\tilde{x}_t, t)(x_t - \tilde{x}_t)^2 + \ldots \qquad \text{(ii)}$$

Insert (i) into (ii) and recursively insert (ii) into itself. Ignore terms above a given order in Ku .This gives an approximation of $u(x_t, t)$ in terms of u, $\partial_x u$, $\partial_x^2 u$ etc. evaluated at the deterministic trajectory $\tilde{x}_t$.

Trajectory approximation ($d = 1$)

Expansion is good up to some Ku - and St -dependent time scale t^*.



Steady state averages ($d = 1$)

Average over $u(\tilde{x}_t, t)$, $\partial_x u(\tilde{x}_t, t)$, $\partial_x^2 u(\tilde{x}_t, t)$,... with known distribution $P(u, \partial_x u, \ldots)$ (Gaussian here) along the deterministic trajectories $\tilde{x}_t$

$$\langle X \rangle_t \equiv \int \mathrm{d}u \mathrm{d}\partial_x u \cdots P(u, \partial_x u, \ldots) X(x_t, t)$$

where X denotes a single particle dynamical quantity, e.g. $\dot{x}$, $\partial_x u(x_t, t)$, ...

For small enough Ku or large enough St the expansion is valid for many correlation times. This allows neglection of the initial configuration x_0, $\dot{x}_0$, $u(0)$, $\partial_x u(0)$,.... for large times. Steady state time averages along trajectories are calculated as

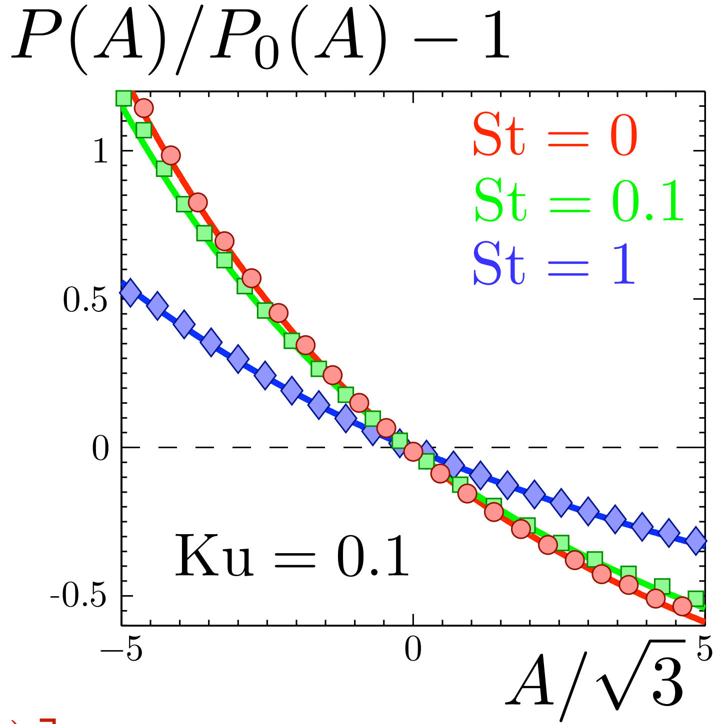
$$\langle X \rangle_\infty = \lim_{T \to \infty} \langle X \rangle_T$$

Preferential distribution of $A = \partial_x u$

The first two moments of A:

$$\langle A \rangle_\infty = -\frac{3\mathrm{Ku}}{1+\mathrm{St}} + \dots$$

$$\langle A^2 \rangle_\infty = 3 + \frac{9\mathrm{Ku}^2(1+3\mathrm{St})}{(1+\mathrm{St})^2(1+2\mathrm{St})} + \dots$$



Calculation of all moments $\langle A^m \rangle_\infty$ gives the distribution along preferential trajectories

$$P(A) = \left[1 - \frac{A\mathrm{Ku}}{1+\mathrm{St}} + \frac{(A^2-3)\mathrm{Ku}^2(1+3\mathrm{St})}{2(1+\mathrm{St})^2(1+2\mathrm{St})}\right] P_0(A)$$

$$P_0(A) = \frac{1}{\sqrt{6\pi}} e^{-A^2/6}$$

D.N.S.
Theory $\mathcal{O}(\mathrm{Ku}^4)$

Trajectory approximation ($F \neq 0$)

Solve equations of motion (dimensionless units)

$$\dot{\boldsymbol{r}} = \mathrm{Ku}\boldsymbol{v} \;\;,\;\; \dot{\boldsymbol{v}} = (\boldsymbol{u}(\boldsymbol{r}_t, t) - \boldsymbol{v})/\mathrm{St} + F\hat{\boldsymbol{g}}$$

implicitly

$$\boldsymbol{r}_t = \tilde{\boldsymbol{r}}_t + \frac{\mathrm{Ku}}{\mathrm{St}} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 e^{-(t_1 - t_2)/\mathrm{St}} \boldsymbol{u}(\boldsymbol{r}_{t_2, t_2})$$

with deterministic part

$$\tilde{\boldsymbol{r}}_t = \boldsymbol{r}_0 + \mathrm{Ku}\boldsymbol{v}_\mathrm{s} t + \mathrm{KuSt}(\boldsymbol{v}_0 - \boldsymbol{v}_\mathrm{s})(1 - e^{-t/\mathrm{St}})\;.$$

Expand the flow $\boldsymbol{u}(\boldsymbol{r}_t, t)$ around $\tilde{\boldsymbol{r}}_t$ and iterate expansion.

Insert the expanded flow into the equation for the velocity gradient matrix $\mathbb{Z} \equiv \boldsymbol{\nabla}\boldsymbol{v}^\mathrm{T}$: $\dot{\mathbb{Z}} = (\boldsymbol{\nabla}\boldsymbol{u}^\mathrm{T}(\boldsymbol{r}_t, t) - \mathbb{Z})/\mathrm{St} - \mathrm{Ku}\mathbb{Z}^2$.

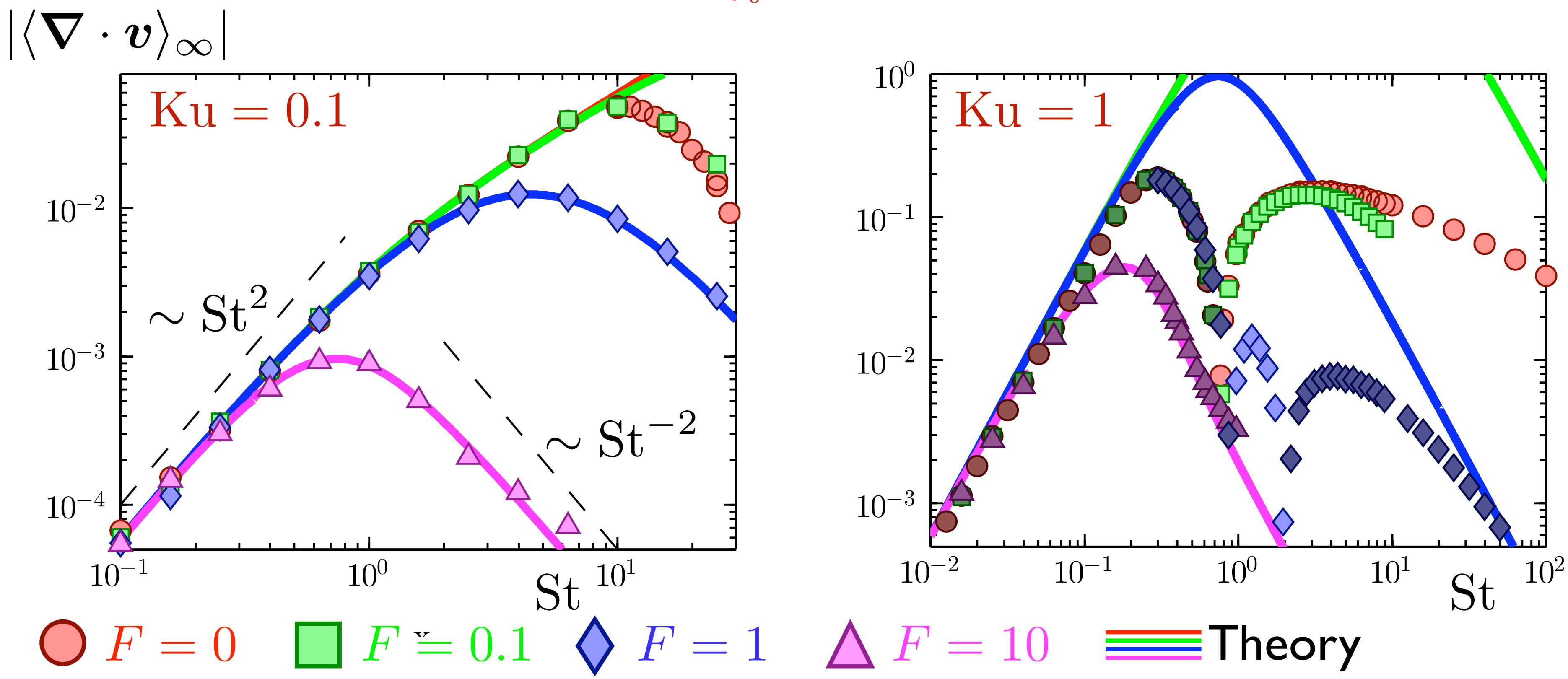
Expand this equation around the $\mathbb{Z}^2$ -term, solve implicitly and iterate to obtain an expansion of $\mathbb{Z}$.

Evaluate average compressibility $\langle \boldsymbol{\nabla} \cdot \boldsymbol{v} \rangle_\infty$ along particle trajectories to determine how areas of closeby particles develop ($\lambda_1 + \lambda_2 = \mathrm{Ku}\langle \boldsymbol{\nabla} \cdot \boldsymbol{v} \rangle_\infty$)

Preferential sampling of $\nabla \cdot \boldsymbol{v}$

We find:

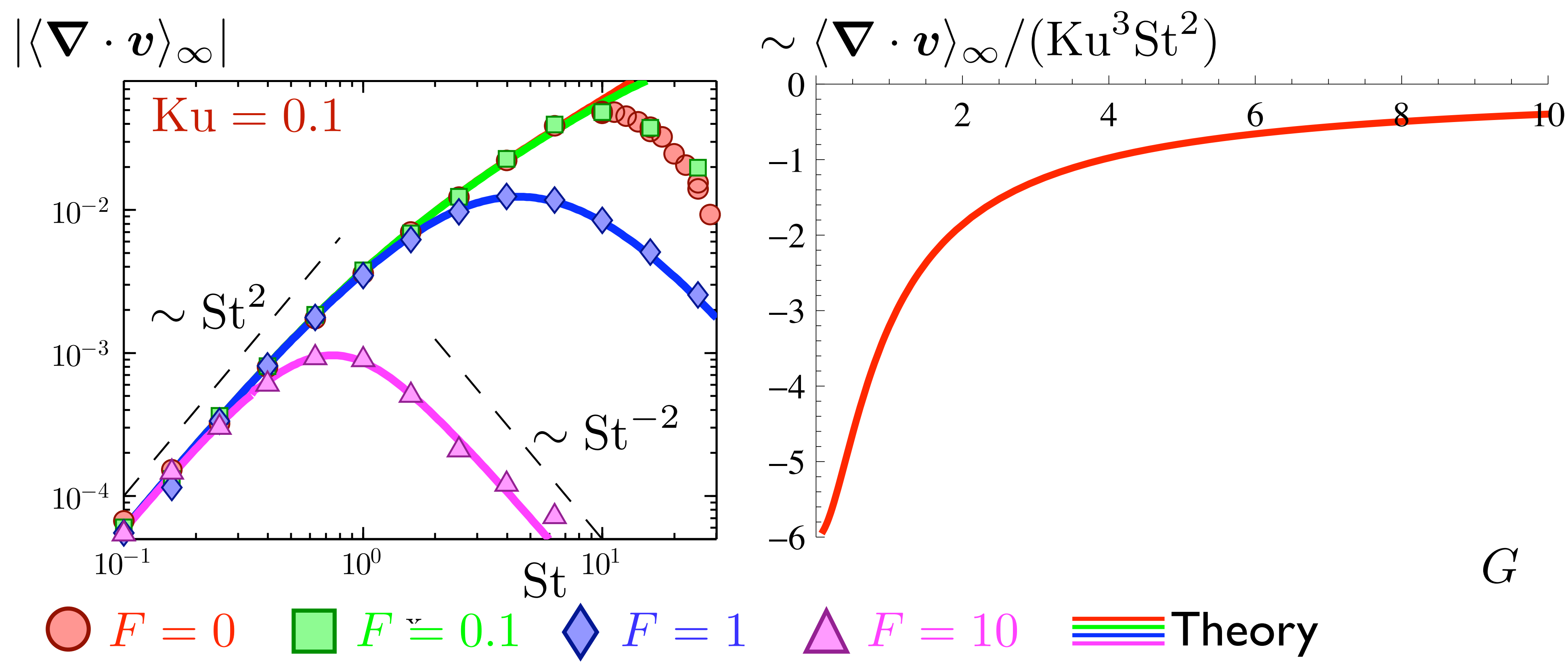
$$\langle \boldsymbol{\nabla}\cdot\boldsymbol{v}\rangle_\infty = \frac{3\mathrm{Ku}^3}{4\mathrm{St}^5G^8}\Big\{2G^2\mathrm{St}^3(5+4\mathrm{St}+3\mathrm{St}^2-G^2\mathrm{St}^2(1+\mathrm{St}))+(1+\mathrm{St})^3(2(1+\mathrm{St})^2-G^2\mathrm{St}^2(\mathrm{St}-3))\mathcal{F}\Big[\frac{1+\mathrm{St}}{\sqrt{2}\mathrm{St}G}\Big]^2$$
$$-\sqrt{2}G\mathrm{St}^2(13+17\mathrm{St}+15\mathrm{St}^2+3\mathrm{St}^3+G^2\mathrm{St}^2(4-\mathrm{St}-3\mathrm{St}^2)+G^4\mathrm{St}^4)\mathcal{F}\Big[\frac{1+\mathrm{St}}{\sqrt{2}\mathrm{St}G}\Big]-4G\mathrm{St}(1+\mathrm{St}^2(2+\mathrm{St}^2+G^2))\mathcal{F}\Big[\frac{1}{G}\Big]$$
$$-2\sqrt{\pi}(1+\mathrm{St}^2)G(-2+\mathrm{St}^2(-2+(-3+\mathrm{St}^2)G^2))\int_0^\infty \mathrm{d}t\exp\Big[G^{-2}-t/\mathrm{St}-G^2t^2/4\Big]\mathrm{erfc}\Big[G^{-1}+Gt/2\Big]\Big\}$$



Preferential sampling of $\boldsymbol{\nabla} \cdot \boldsymbol{v}$

Small St: $\langle \boldsymbol{\nabla} \cdot \boldsymbol{v} \rangle_\infty \sim 3\mathrm{Ku}^3\mathrm{St}^2(4G - 6G^3 - (4 - 4G^2 + 3G^4)\mathcal{F}[G^{-1}])/(4G^5)$

As $G \to 0$ the Maxey result is recovered $\langle \boldsymbol{\nabla} \cdot \boldsymbol{v} \rangle_\infty \sim -6\mathrm{Ku}^3\mathrm{St}^2$

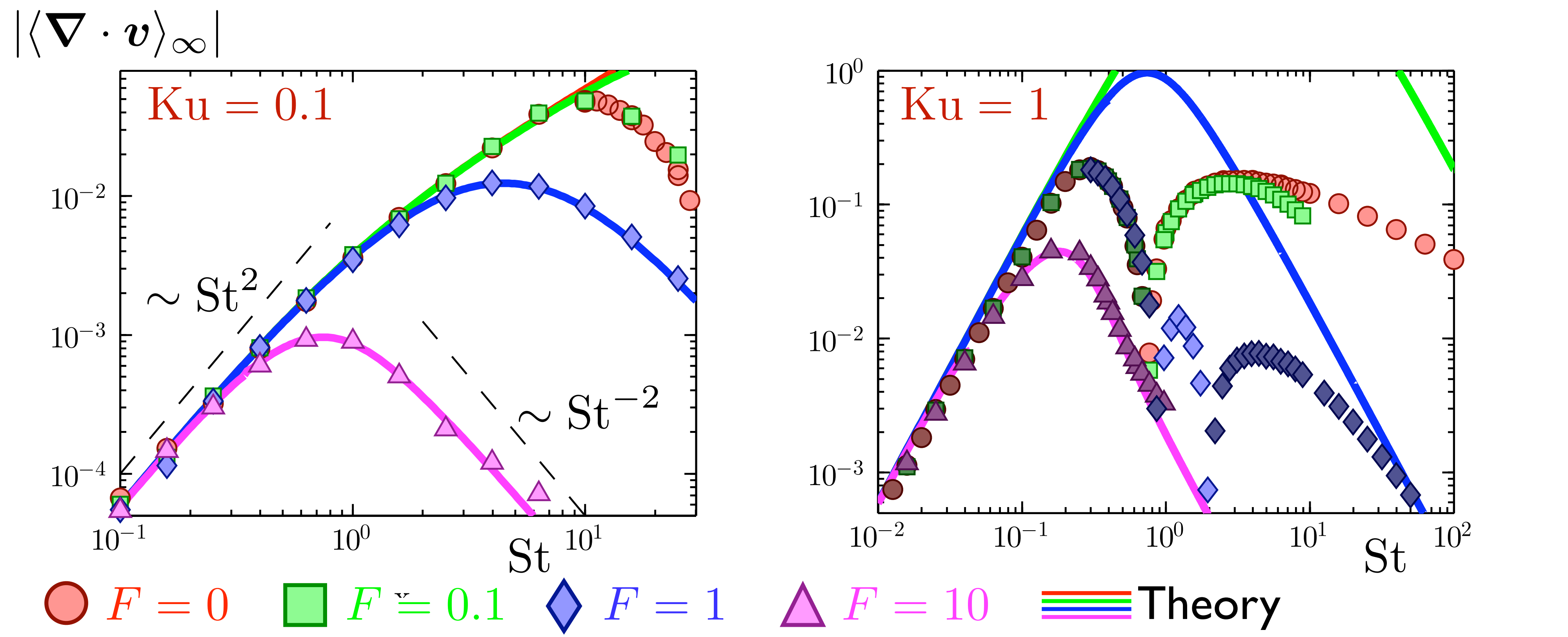


Preferential sampling of $\nabla \cdot \boldsymbol{v}$

Small G : $\langle \nabla \cdot \boldsymbol{v} \rangle_\infty \sim -6\mathrm{Ku}^3\mathrm{St}^2 \dfrac{1+3\mathrm{St}+\mathrm{St}^2}{(1+\mathrm{St})^3} + 9\mathrm{Ku}^3 G^2 \mathrm{St}^2 \dfrac{1+5\mathrm{St}+12\mathrm{St}^2+20\mathrm{St}^3+4\mathrm{St}^4}{(1+\mathrm{St})^5}$

As $G \to 0$ earlier results are recovered

Gustavsson, Mehlig EPL **96** (2011)

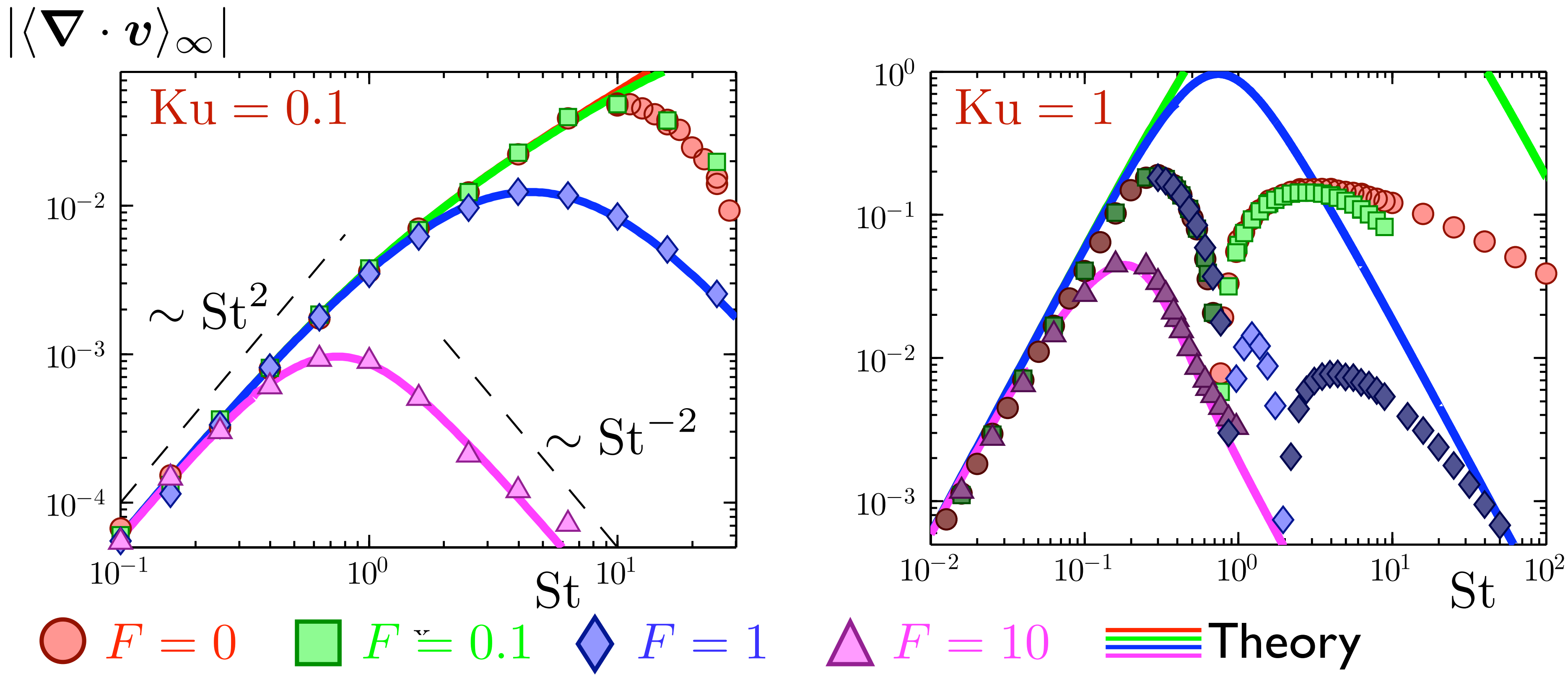


Preferential sampling of $\nabla \cdot v$

Large St , G : $\langle \nabla \cdot v \rangle_\infty \sim -3\mathrm{Ku}^3\mathrm{St}\sqrt{2\pi}/(4G^3)$

Same parameter-dependence as the Langevin model:

$$\mathrm{KuSt}\langle \nabla' \cdot v' \rangle_\infty \sim -3\sqrt{2\pi}/4[\mathrm{Ku}^2\mathrm{St}/G^{3/2}]^2$$



Maximal Lyapunov exponent λ_1

Similar expansion for λ_1 using

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\boldsymbol{R}_t|}{|\boldsymbol{R}_0|} = \lim_{t\to\infty} \frac{\mathrm{Ku}}{t} \int_0^t \mathrm{d}t' \hat{\boldsymbol{R}}_{t'}^{\mathrm{T}} \mathbb{Z}_{t'} \hat{\boldsymbol{R}}_{t'}$$

with $\hat{\boldsymbol{R}}_t \equiv \boldsymbol{R}_t/|\boldsymbol{R}_t|$ gives to lowest order in Ku

$$\lambda_1 = \frac{\mathrm{Ku}^2}{2G^5}\Big[-G^3 + G(1+11G^2)(\hat{\boldsymbol{R}}_0\cdot\hat{\boldsymbol{g}})^2 - 2G(1+5G^2)(\hat{\boldsymbol{R}}_0\cdot\hat{\boldsymbol{g}})^4$$
$$+\frac{1}{\sqrt{2}}\Big\{G^2(1+3G^2)-(1+12G^2+9G^4)(\hat{\boldsymbol{R}}_0\cdot\hat{\boldsymbol{g}})^2+2(1+6G^2+3G^4)(\hat{\boldsymbol{R}}_0\cdot\hat{\boldsymbol{g}})^4\Big\}\mathcal{F}\Big[\frac{1}{\sqrt{2}G}\Big]\Big]$$

depends on the unit vector $\hat{\boldsymbol{R}}_0$ for a small initial separation between two particles.

Find λ_1 by averaging $(\hat{\boldsymbol{R}}_0\cdot\hat{\boldsymbol{g}})^{2p}$ with $p = 1, 2, \ldots$ using steady-state averages $\langle(\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{g}})^{2p}\rangle_\infty$.

Complication: Steady-state averages $\langle(\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{g}})^{2p}\rangle_\infty$ in turn depend on $(\hat{\boldsymbol{R}}_0\cdot\hat{\boldsymbol{g}})^{2p}$ and contain secular terms.

Preferential alignment

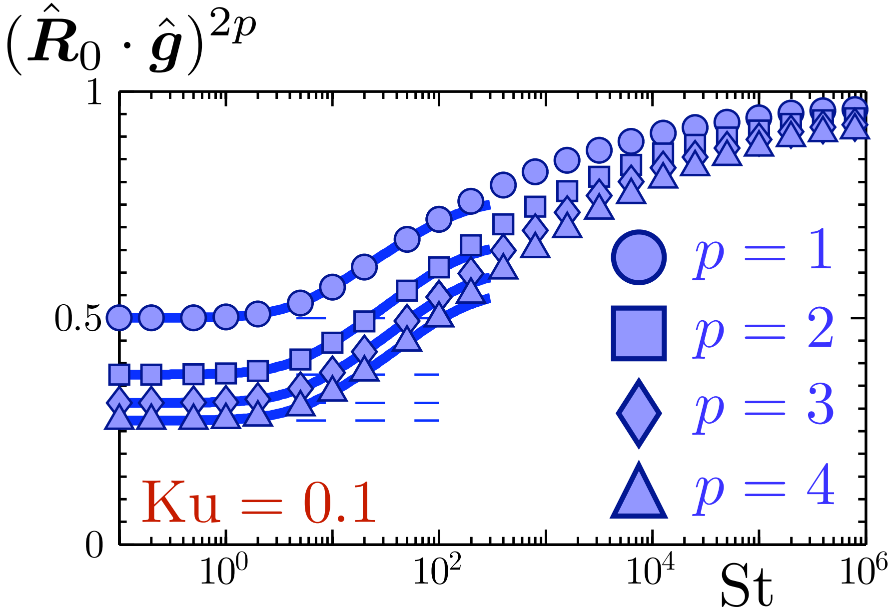
Self-consistency solution to remove the secular terms gives recursion relations for the moments $\langle(\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{g}})^{2p}\rangle_\infty$.

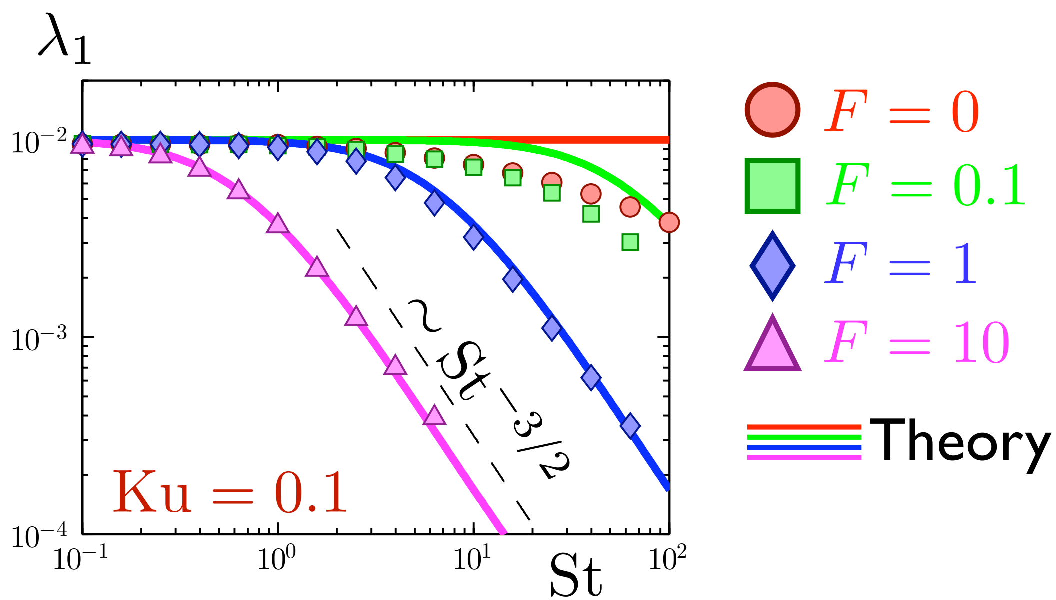
These recursions can be solved if series expanded in small $G = \mathrm{Ku}\mathrm{St}F$.

We find

$$\langle(\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{g}})^{2p}\rangle_\infty = \frac{(2p-1)!!}{2^p p!}\left[1+\frac{pG^2}{p+1}-\frac{p(41+19p)G^4}{4(p+1)(p+2)}+\ldots\right]$$

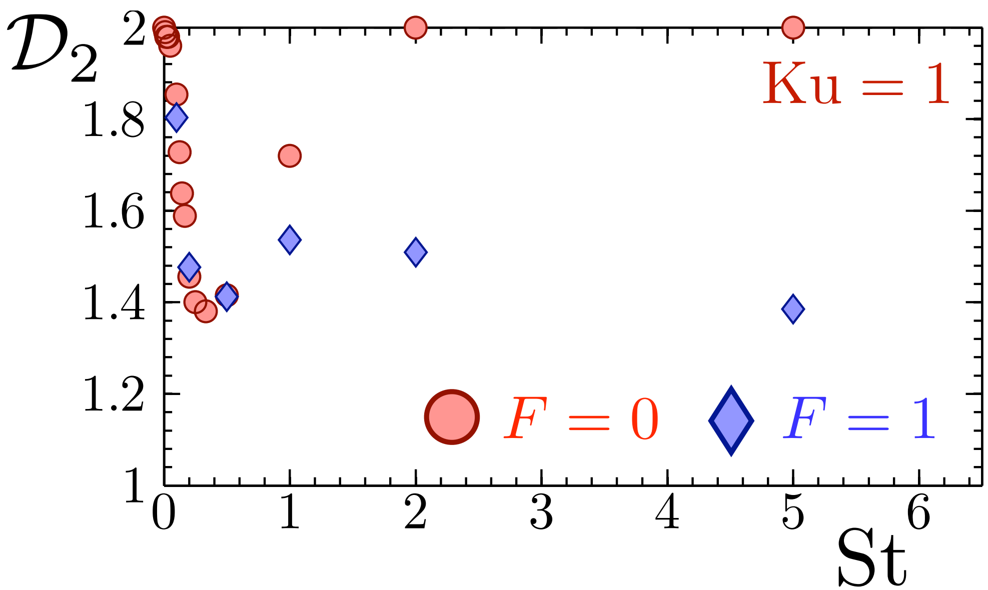
Padé-Borel resum this series to find the theory plotted below





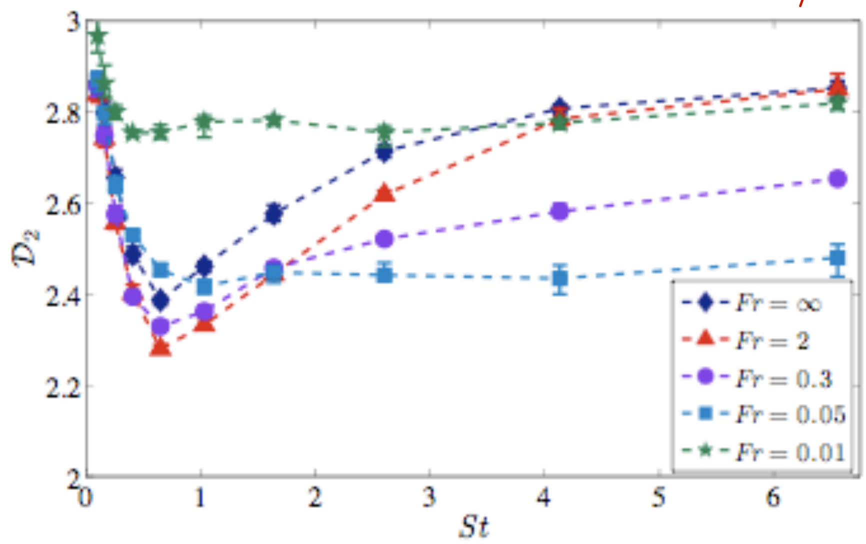
Comparison to turbulence

Comparison of random-flow model ($d = 2$) to results from DNS. Correlation dimension $\mathcal{D}_2$ defined by scaling $\rho(R) \sim R^{\mathcal{D}_2 - 1}$ of distribution of distances $\rho(R)$ for small distances R.



Random-flow model

$\mathrm{Fr} = 1/F$



DNS

J. Bec, H. Homann, S. Ray arXiv:1401.1306

Conclusions

Inertial response to flow fluctuations and the effect of gravity are not additive.

Small St : Gravity reduces clustering because correlations between particles and flow structures are weakened.

Large St : Gravity may increase clustering significantly due to multiplicative amplification.

Gravity introduces an anistropy in the spatial distribution of close-by particles. Particle separations align with $\pm\hat{\boldsymbol{g}}$.